Aisle Configurations for Unit-Load Warehouses

Kevin R. Gue
Department of Industrial & Systems Engineering
Auburn University
Auburn, Alabama 36849
kevin.gue@auburn.edu

Russell D. Meller
Department of Industrial Engineering
University of Arkansas
Fayetteville, Arkansas 72701
rmeller@uark.edu

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Abstract

Unit-load warehouses are used to store items—typically pallets—that can be stowed or retrieved in a single trip. In the traditional, ubiquitous design, storage racks are arranged to create parallel picking aisles, which force workers to travel rectilinear distances to picking locations. We consider the problem of arranging aisles in new ways to reduce the cost of travel within these warehouses. Our models produce alternative designs with piecewise diagonal cross aisles, and with picking aisles that are not parallel. One of the designs promises to reduce distances that workers travel by more than 20 percent for warehouses of reasonable size. We also develop a theoretical bound that shows that this design is close to optimal.

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1 Warehouse Design

A distribution center (DC) is a complex collection of people, space and machines, designed to store and deliver goods when requested. Traditionally, the storage role has been emphasized, and the design goal for distribution centers has been to reduce cost by minimizing the size of the facility. But increasingly, distribution centers are being measured by performance on quality and service measures (Maltz and DeHoratius, 2004), which suggests that firms realize the more strategic role distribution centers can play in competition. In a sense, distribution centers have become the “factories of the service economy.”

Maltz and DeHoratius report that modern distribution centers tend to be larger and more capable of performing non-traditional functions than their older counterparts. The reasons for this are many, including higher cost of land near major metropolitan areas (which leads firms to have fewer DCs to perform the same functions), the desire to reduce safety stocks by consolidating inventories into fewer locations, and the rise of third-party DCs, which may house the products of multiple distributors. A recent Department of Energy study reports that there are more than 15,000 warehouses larger than 100,000 square feet and nearly 1,000 larger than 500,000 square feet in the United States alone (Energy Information Administration, 2006).

As the size of a distribution center increases, capital and operating costs also increase: capital expenses because there are more storage racks, lift trucks, and so on; and operating costs because there are more workers, who must travel greater distances to store and retrieve items. Among operating costs, worker travel is very significant in most distribution centers. One DC with which we have worked employs more than 90 workers in 3 shifts in a pallet picking area alone, and the majority of their time is spent traveling. Naturally, as the size of a DC increases, we can expect travel costs to increase as well.

A distribution center consists of several component subsystems, including receiving, storage, order picking, and shipping. Perhaps the most common building block in these systems is the pallet storage area, which consists of storage racks and aisles between them. In the academic literature this area is commonly called a “warehouse,” and we adopt that terminol-
ogy here. (The term “warehouse” can also refer to the entire distribution center.) Because almost all products are received and stored—at least temporarily—in pallet quantities, pallet warehouses tend to consume the majority of space within a distribution center.

In practice, warehouses within distribution centers are of two types—case picking and unit-load. In a case picking warehouse, workers circumnavigate aisles with picking carts (or perhaps ride a forklift, with an empty pallet) and build orders by picking items or cases from the stored pallets. Ratliff and Rosenthal (1983) showed that routing a worker to pick several items from a rectangular, case-picking warehouse (Figure 1, left) is a solvable case of the Traveling Salesman Problem. When warehouses of this configuration get large, picking tours tend to be long because workers are forced to one end of the warehouse or the other when moving between aisles. Roodbergen and de Koster (2001) extended Ratliff and Rosenthal’s results to the case of rectangular picking areas with one or more cross aisles (Figure 1, right). They found that having a cross aisle is not beneficial when retrieving a single item, because the cross aisle effectively moves half the pallets farther from the pickup-and-deposit (P&D) point at the bottom. For “reasonably-sized” pick-lists, however, a cross aisle allows shorter tours. For very large pick lists, which are not uncommon in industry, the cross aisle again confers a dis-advantage because nearly every aisle is traversed in its entirety and the cross aisle effectively makes the picking aisles longer. Vaughan and Petersen (1999) used heuristic routing techniques to determine the number of cross aisles in a picking area.

In a unit-load warehouse, items are stowed and retrieved in pallet quantities, or unit-load quantities, and each stow or pick is for a single item. Unit-load warehouses are used in at least two ways in a distribution center: (1) as an order picking area, when products are received and shipped in pallet quantities (distributors of groceries or appliances are two examples); and (2) as reserve areas that replenish fast-pick areas (Bartholdi and Hackman, 2006). For example, a common arrangement is to have a fast-pick “module,” consisting of gravity-fed, carton flow rack, replenished from a pallet reserve area. Order pickers build detailed orders from the flow rack, while workers on lift trucks replenish product in the flow rack from the reserve pallet area. Unit-load operations are also common in crossdocking,
Figure 1: Configurations of typical warehouses in industry. On the left, a unit-load warehouse; either design may constitute a case-picking warehouse, depending on the size. The cross aisle in the warehouse on the right allows for shorter tours when gathering multiple items.

where pallets are stored briefly before being loaded onto outbound trucks.

Unit-load warehouses in industry are typically comprised of single- or double-deep pallet racks arranged in parallel picking aisles, as in Figure 1. In keeping with the results of Roodbergen and de Koster (2001), the established orthodoxy is that strictly unit-load warehouses should not include a cross aisle. This traditional design has been preferred because the storage area is as small as possible, which confers two advantages: (1) construction costs are lower, and (2) smaller picking areas are believed to reduce travel distances and therefore travel costs. (We will show the latter belief to be ill-founded.)

Francis (1967) investigated rectangular warehouse shapes to minimize picking and construction costs. He assumed rectilinear travel paths, which “presupposes that there is an orthogonal network of aisles running parallel to the $x$ and $y$ axes.” Bassan et al. (1980) developed models to determine when it is best to align picking aisles horizontally or vertically in a warehouse, but they assume the traditional structure of Figure 1, with all picking aisles parallel. Berry (1968) noticed that floor-stored pallets should be arranged in lanes with
different depths, based on demand characteristics for the sku, and that different lane depths can be arranged to form “diagonal gangways” in the storage space. He did not explore the implications of this observation.

There seem to be no papers, with the possible exception of Berry (1968), that address the internal design of a storage area; rather, the existing literature uniformly addresses the operation of a presupposed one, which is always some variation of the designs in Figure 1. In fact, our experience suggests that Figure 1 may represent all rack-and-aisle storage areas in industry and academic research—for both unit-load and case-picking operations.

The focus of our work is unit-load warehouses, all of which, we believe, conform to two Unspoken Design Rules:

1. Picking aisles must be straight, and they must be parallel.

2. If present, cross aisles must be straight, and they must meet picking aisles at right angles.

Given these two rules and a rectangular warehouse space, the designs in Figure 1 are probably the only options. Yet neither rule is necessary from a construction point of view, and, as we are about to show, both tend to reduce productivity. This leads us to ask,

*How should cross aisles and picking aisles be arranged to minimize the expected time to pick in a unit-load warehouse?*

Our contribution in this work is both theoretical and practical. Theoretically, we introduce a problem fundamental to the design of warehouses which, we believe, has been overlooked by the research community. Although our models are relatively straightforward, the solutions suggest two aisle configurations that promise significant productivity gains. One of these contradicts the established orthodoxy that inserting a cross aisle is not beneficial in a unit-load warehouse. We also propose a simple bound on the potential improvement of any warehouse design, and show that one of our designs provides nearly all of this potential improvement. Practically, the warehouse designs are easy to implement, and should allow distributors to “leverage” the time savings in two ways: by reducing the labor assigned to
complete a set of picks, or by using existing labor to complete the picks more quickly. The former reduces costs, the latter improves service; which is preferred depends on the goals of the distributor.

In the next two sections we consider two design problems. First, we build a model to find the optimal cross aisle for a unit-load warehouse with parallel picking aisles. We show that, contrary to the result of inserting a straight cross aisle in a unit-load warehouse, inserting the cross aisle defined by this problem yields a significant reduction in expected travel distance to make a pick. This design is amenable to retrofitting an existing facility. Second, we propose warehouse designs with picking aisles that are not parallel. We combine this flexibility with a V-shaped cross aisle to build a warehouse with a “fishbone” aisle structure. The expected retrieval times for these designs are as much as 20 percent lower than for equivalent traditional warehouses. This design is better-suited to greenfield designs. In Section 4 we propose a theoretical bound for improvement of any aisle design over a traditional warehouse, and show that the improvement offered by fishbone aisles is very close to the bound. We address implementation issues in the final section.

2 An Optimal Cross Aisle

In this first model, we insert a single cross aisle into the picking space and search for a shape that improves expected travel distance to a pick. Although we can imagine inserting more than one cross aisle, doing so further reduces storage density, which is often a significant issue in unit-load warehouses. We model the cross aisle as a set of connected line segments, where each segment connects adjacent picking aisles (see Figure 2). The objective is to find points of intersection between the cross aisle and the picking aisles that minimize the expected distance to make a pick or stow. We assume that all picking aisles are parallel (Unspoken Design Rule 1), but do not require that the cross aisle be orthogonal to the picking aisles, or even that it be straight (Unspoken Design Rule 2). We also assume a “single command cycle,” in which a worker drives unloaded to the requested item and returns loaded directly
Figure 2: The continuous optimal cross aisle warehouse model, with nominal rack locations shown on the left side. Workers may travel along the bottom cross aisle or along the middle cross aisle segments. Gray boxes around the $b_i$ points indicate regions where no picking takes place due to the width of the cross aisle. Variables $d_i$ and $a$ represent lengths of their respective aisle segments. Each picking aisle has a point $q_i$ at which a worker is indifferent to traveling along the cross aisle or the bottom aisle.

2.1 Model

Consider a set of picking aisles, each having a continuous distribution of picking activity within the aisle. For large warehouses this is a reasonable assumption because a picking aisle is often more than 50 picking locations long, and we are interested only in the expected distance to store or retrieve a pallet. Moreover, we assume a uniform distribution of picking activity within and among all picking aisles. This is approximately equivalent to the “closest open location” policy for assigning incoming pallets to a warehouse (as opposed to considering the activity level of the item when making this assignment). In the literature this is referred to as *randomized storage*. Randomized storage is very common in unit-load warehouses because it tends to maximize space utilization (Tompkins et al., 2003), and warehouse management systems can easily direct workers to necessary locations.
Because the picking space is symmetric about the P&D point, we can focus our modeling on a picking half-space only. The goal is to construct a cross aisle in the half-space that minimizes the expected distance to make a pick. The P&D point is located in the bottom left corner of the half-space, when viewing it from the top. The cross aisle intersects each picking aisle \( i \) at a point \( b_i \), including picking aisle zero, which ends at the P&D point. A line connecting the \( b_i \)'s forms the cross aisle (see Figure 2).

We assume the cross aisle consumes \( 2w \) of each picking aisle, and so inserting it causes the storage space to be larger than it otherwise would be. Notice that if the cross aisle does not intersect picking aisles at a right angle, the width of the cross aisle is less than \( 2w \)—in fact, the width depends on the choice of the \( b_i \)'s. We ignore this detail, and assume a judicious choice of \( w \), so that the resulting cross aisle is sufficiently wide. In practice, one could easily adjust the choice of \( w \) and run the model again to achieve a satisfactory width.

Assume the picking half-space has \( n + 1 \) picking aisles, each with height \( h \), at a distance \( a \) apart (aisle zero extends directly upward from the P&D point). In practice, the picking half-space would include a cross aisle of width \( 2w' \) at the bottom of the picking space and perpendicular to the picking aisles. Because every pick regardless of its location would include \( w' \) travel distance across this aisle, we do not consider it in the optimization model. We do include this detail in numerical results later in the paper.

By Pythagoras’ Theorem, the portion of the cross aisle between picking aisles \( i - 1 \) and \( i \) has length
\[
d_i = \sqrt{a^2 + (b_i - b_{i-1})^2}.
\]  
We do not require \( b_i > b_{i-1} \), and so the model is free to construct a cross aisle taking on any continuous path.

Let \( q_i < b_i \) be the point along picking aisle \( i \) at which a worker is indifferent to traveling either along the bottom cross aisle (hereafter, bottom-aisle) and up the picking aisle, or along the cross aisle and down (see Figure 2). That is,
\[
ia + q_i = b_0 + \sum_{k=1}^{i} d_k + (b_i - q_i),
\]
or,

\[ q_i = \frac{1}{2} \left( b_0 + \sum_{k=1}^{i} d_k + b_i - ia \right). \]  

(2)

We require \( q_0 = 0 \), by definition.

Let \( C_i(x) \) be the travel distance (hereafter, travel cost) to pick an item at distance \( x \) from the bottom of aisle \( i \), and assume the cross aisle consumes distance \( 2w \) of each picking aisle. We divide the picking aisle into three regions, corresponding to different travel paths for shortest retrieval. For \( x < q_i \), it is best to travel along the bottom-aisle and up. For \( q_i \leq x \leq b_i \), travel is along the cross aisle and down; for \( x \geq b_i \), travel is along the cross aisle and then up. We require \( w \leq b_i \leq h - w \) to ensure that each picking aisle retains \( h - 2w \) of picking length.

Let \( f_X(x) \) be the probability density function of a pick at position \( x \). In this paper we assume a uniform pick density, so \( f_X(x) = \frac{1}{h-2w} \). The expected travel cost for a pick in aisle \( i \geq 1 \) is

\[
E[C_i] = \int_0^h C_i(x) f_X(x) \, dx = \frac{1}{h-2w} \int_0^h C_i(x) \, dx
\]

\[
= \frac{1}{h-2w} \left[ \int_0^{q_i} C_i(x) \, dx + \int_{q_i}^{b_i-w} C_i(x) \, dx + \int_{b_i+w}^h C_i(x) \, dx \right]
\]

\[
= \frac{1}{h-2w} \left[ \int_0^{q_i} (ia + x) \, dx + \int_{q_i}^{b_i-w} \left( b_0 + \sum_{k=1}^{i} d_k + b_i - x \right) \, dx + \int_{b_i+w}^h \left( b_0 + \sum_{k=1}^{i} d_k + x - b_i \right) \, dx \right]
\]

\[
= \frac{1}{h-2w} \left[ q_i \left[ ia + \frac{1}{2} q_i \right] + (b_i - w - q_i) \left( b_0 + \sum_{k=1}^{i} d_k + \frac{1}{2} (b_i + w - q_i) \right) + (h - b_i - w) \left[ b_0 + \sum_{k=1}^{i} d_k + \frac{1}{2} (h - b_i + w) \right] \right],
\]

where \( d_i \) and \( q_i \) are given in (1) and (2), respectively. The expression has the following interpretation: if \( x \leq q_i \), the expected travel is along the bottom \( ia \) units, then up \( q_i/2 \) units; if \( q_i \leq x \leq b_i - w \), expected travel is up to \( b_0 \), along the cross aisle \( \sum_{k=1}^{i} d_k \) units, then down \( w + (b_i - w - q_i)/2 \) units; if \( x \geq b_i \), travel is up to \( b_0 \), along the cross aisle \( \sum_{k=1}^{i} d_k \) units, then up \( w + (h - b_i - w)/2 \) units. Each expected travel distance is weighted by the
length of the appropriate region.

The expected travel cost to pick an item in aisle zero is different, because there is no travel along the cross aisle. For \( b_0 \geq w \), we have two possible regions—one below the cross aisle, and one above. For a uniform picking density, the expected travel cost is simply the weighted sum of costs to make a pick from the center of each region.

\[
E[C_0] = \frac{b_0 - w}{h - 2w} \left( \frac{1}{2}(b_0 - w) \right) + \frac{h - b_0 - w}{h - 2w} \left( b_0 + w + \frac{1}{2}(h - b_0 - w) \right) = \frac{h^2 - 4b_0w}{2(h - 2w)}.
\]

The expected travel cost for a pick in the full picking space includes a term for a pick in aisle zero, plus (due to symmetry) two times the terms for remaining aisles in the half-space,

\[
E[C] = p_0E[C_0] + 2 \sum_{i=1}^{n} p_iE[C_i],
\]

where \( p_i = 1/(2n + 1) \) is the probability of the pick being in aisle \( i \), for \( i = 0, \ldots, n \).

One final detail: if \( q_i > b_i - w \), the cost expression \( E[C_i] \) is malformed, and so we require \( q_i \leq b_i - w \). This constraint is tight only if \( w \) is large with respect to \( h \), or if \( b_i \) is very small, neither of which is true for practical problems. We can restate this constraint as \( b_i \geq w + q_i \), which is stronger than the previous constraint \( b_i \geq w \). The optimization problem is to choose \( b = \{b_0, \ldots, b_n\} \) such that \( E[C] \) is minimized, subject to \( w + q_i \leq b_i \leq h - w \), for all \( i \).

Note that we have chosen not to formulate the problem with a network model, with arcs connecting points between adjacent aisles and a “shortest path” representing the cross aisle. Doing so would have been appealing for two reasons: (1) it might be easy to solve to optimality, and (2) a warehouse full of pallet rack imposes discrete locations for aisles anyway. Unfortunately, this approach is not possible because we cannot assign \( a \ priori \) costs to arcs that would make up the cross aisle. For example, the cost of an arc should represent the level of flow along the arc, but that flow cannot be specified until the flows along the cross aisle beyond that arc are specified. Similarly, the flow along an arc cannot be specified until the aisle arcs before it are specified, because they determine how costly it is to get to that arc. Because costing the arcs requires that the entire cross aisle be specified \( a \ priori \), the problem does not lend itself to constructive graph algorithms or to a sequential optimization technique such as dynamic programming.
We use instead numerical, non-linear optimization techniques to solve our continuous formulation, and it solves easily. We applied a number of standard algorithms and found no measurable difference among them. Although we cannot prove optimality for our solutions, their structure is intuitive and likely close to optimal. Moreover, it is likely that, in practice, any solution would have to be adjusted anyway to account for the discrete nature of pallet locations.

2.2 Structure of Optimal Solutions

There is a value of $w$ greater than which we would choose not to have a cross aisle, because its presence pushes some locations so far from the P&D point that the additional distance overwhelms the advantage of having the cross aisle. Unfortunately, that “critical value” of $w$ depends on the $b_i$’s, which in turn depend on $w$, and so we are unable to compute it directly. Fortunately, the model handles this difficulty implicitly: if $w$ is greater than the critical value, the cross aisle is along the top of the picking space, effectively making it no cross aisle at all.

Analytical results on the structure of optimal solutions are difficult because of the interdependency between the points of intersection $b_i$ and the critical value of $w$ for which we should choose not to have a cross aisle. However, if $w = 0$, we can develop some results, to which practical problems (with realistic values for $w$) seem to conform. Proofs of the following results are in the Appendix.

**Proposition 1** If $w = 0$, an optimal solution has $b_0 = 0$; that is, the optimal cross aisle begins at the P&D point.

The result agrees with our intuition that the purpose of a cross aisle is to get the worker into the interior of the picking space as quickly as possible. We can also prove that the cross aisle increases in height as it moves away from the P&D point.

**Proposition 2** If $w = 0$, an optimal solution has $b_i \geq b_{i-1}$, for all $i \geq 1$. 
Finally, there is an upper bound on any $b_i$,

**Proposition 3** If $w = 0$, the optimal point of intersection $b_i$ satisfies $b_i \leq \frac{2}{3}h + \frac{1}{3}(\sum_{k=1}^{i} d_k - ia)$.

The bound is not tight for small warehouses, but is quite tight for large ones, as we show by example in the next section.

These results suggest that, for a warehouse with $w = 0$, the optimal cross aisle begins at the P&D point and increases steadily into each picking half-space, intersecting picking aisles at points that approach a value slightly greater than $2h/3$. Cross aisles in all the designs we have constructed (with realistic values of $w$ and other parameters) exhibit this shape, which we call the “V-shaped” cross aisle design (see Figure 3 for two examples).

**Examples.** Assume that with appropriate clearances a pallet occupies a $4' \times 4'$ square, and picking and cross aisles are 10 feet wide. These values will vary in practice, depending on pallet sizes, type of picking vehicles, and so on, but our choices represent the majority of warehouses in industry (Tompkins et al., 2003). In what follows, our unit of measure is a pallet; e.g., aisles are 2.5 pallets wide, and perhaps 50–100 pallets long.

Figure 3 shows solutions to our model for two warehouses, one with picking aisles 100 pallets long, the other 50. The upper design has 21 picking aisles, the lower has 41 aisles. Distance between centers of picking aisles is equivalent to 4.5 pallets (about 18 feet). Dots in the figure correspond to $b_i$ values from the model. The upper design has expected travel distance to a pick 10.0 percent lower than a traditional warehouse (Figure 1, left) with the same number and total length of picking aisles. The lower design offers an advantage of 8.4 percent. These comparisons keep the picking positions constant, but increase the size of the warehouse to account for inserting a cross aisle. The main insight behind these designs is that a cross aisle that cuts diagonally across the picking aisles affords a “Euclidean advantage,” which allows workers to get to most picking locations more quickly. Workers would prefer strict rectilinear travel only to locations near the bottom of the warehouse. In the bottom design in Figure 3, the right- and leftmost $b_i$ values are approximately 68.5% of $h$, slightly
Figure 3: Results of the model for two random-storage, unit-load warehouses with picking aisles 100 and 50 pallets long. Expected travel distance for a pick in the upper design is 10.0% less than in an equivalent traditional warehouse (although the space is 4.1% larger). The lower design offers an 8.4% advantage (with an 8.5% increase in space). Black squares in each design indicate $b_i$ values.
Figure 4: Percent improvement for warehouses with a V-shaped cross aisle, compared to a traditional warehouse. The solid line represents improvement for a warehouse with $h = 100$; the long-dashed line represents 75; the short-dashed line 50. Typical warehouses in industry have 20–40 aisles.

greater than $2h/3$, but still less than the bound in Proposition 3. (Remember: the bound holds strictly only when $w = 0$, which is not the case here.)

Figure 4 illustrates the advantage of V-shaped cross aisles for warehouses of several sizes. In general, as the number of aisles increases, the advantage of inserting a cross aisle increases, until the warehouse is so large that travel to outlying points is dominated by travel to the picking aisle rather than within the picking aisle. For warehouses this large, travel along the cross aisle is almost equal to travel along the bottom aisle, and so little advantage is gained from having the cross aisle. Unit-load warehouses in industry typically have 20–40 aisles, which seems to be the range for which this design confers the greatest advantage. Warehouses with longer picking aisles stand to benefit more from V-shaped picking aisles than do those with short picking aisles.
3 Fishbone Aisles

Next, we relax Unspoken Design Rule 2, which requires that all picking aisles be parallel. In the most general case, picking aisles can take on any angle, but here we consider only vertical and horizontal orientations. We also restrict ourselves to a cross aisle that extends in two diagonals away from the P&D point. We show in the next section—through a theoretical lower bound on expected travel cost—that these constraints do not keep us far from an optimal solution.

3.1 Model

Because the picking space is symmetric, we focus again on a picking half-space. Consider a half-space with \( n \) vertical aisles, each of height \( h \) and separated by distance \( a \). For now, we do not include an aisle extending from the P&D point. Using insights from the previous model, we insert a straight, diagonal cross aisle passing through the P&D point. Let \( 0 \leq b \leq h - w \) be the point of intersection of the cross aisle with the rightmost (\( n \)-th) picking aisle. Using the P&D point as the origin in a coordinate system, we see that the cross aisle has slope \( m = b/na \). The case of the diagonal cross aisle having a greater slope, and therefore reaching the top of the warehouse before reaching the rightmost aisle, is easily handled by “inverting” the space (thereby making the new height equal to \( na \)) and re-solving. The cross aisle consumes distance \( w \) of each picking aisle (see Figure 5).

We break the total expected travel cost into three components: aisle zero, which extends upward from the P&D point, upper aisles (with respect to the cross aisle), and lower aisles. We assume again that picking activity is distributed uniformly among and within all aisles; therefore, expected cost to make a pick in aisle zero is simply

\[
E[C_0] = \int_{w}^{h} x f_X(x) dx = \frac{1}{h - w} \int_{w}^{h} x dx = \frac{(h + w)}{2}.
\]

The travel cost for picking in upper aisle \( i \), \( C_i^u(x) \), is the required distance along the diagonal aisle, plus the travel up to point \( x \). Between each vertical picking aisle, the cross
Figure 5: A continuous model for warehouses with fishbone aisles, with nominal rack locations shown on the left. Gray boxes at the ends of picking aisles indicate regions where no picking takes place due to the width of the cross aisle. Figures on the bottom illustrate changes in aisle structure with different values of the parameter $b$. 
aisle has distance (see Figure 5),
\[ d_v = \sqrt{a^2 + (am)^2} = a\sqrt{1 + m^2}. \]

The expected travel cost of a pick in an upper aisle is,
\[
E[C_i^u] = \int_{mia+w}^{h} C_i^u(x)f_X(x)\,dx = \frac{1}{h-mia-w} \int_{mia+w}^{h} (id_v + x - mia)\,dx = id_v + \frac{1}{2}(h-mia+w). \tag{3}
\]

For the lower picking area we have,
\[ d_h = \sqrt{a^2 + (a/m)^2} = a\sqrt{1 + (1/m)^2}. \]

Lower aisle \(j\) begins at position \(aj/m\) and ends at position \(an\), so
\[
E[C_j^h] = \int_{aj/m+w}^{an} C_j^h(x)f_X(x)\,dx = \frac{1}{an-aj/m-w} \int_{aj/m+w}^{an} (jd_h + x - aj/m)\,dx = jd_h + \frac{1}{2}(an-aj/m+w). \tag{4}
\]

Both (3) and (4) agree with the intuition that expected travel cost for a pick in an aisle is travel to the picking aisle plus half the length of the aisle.

Notice from Figure 5 that there are values of \(b\) for which the distance between the center of the cross aisle and the end of a potential vertical or horizontal aisle is less than \(w\); that is, there would be effectively no picking aisle at all. For a horizontal aisle, this happens when \(b\) is slightly greater than a multiple of \(a\). For a vertical aisle, if \(b = h - w\) and \(h\) is small with respect to \(na\), the length of the rightmost vertical aisles could be less than \(w\). Computationally, an aisle shorter than \(w\) causes (3) or (4) to be negative, and so we define the length of a vertical aisle \(l_i^v = \max\{h - mia - w, 0\}\) and for a horizontal aisle \(l_j^h = \max\{an - aj/m - w, 0\}\). Note that the number of aisles having non-zero length in both the upper and lower regions is a function of parameter \(b\).
Let $U$ be the set of picking aisles in the upper region (perhaps with length zero), and $L$ the set in the lower region. For a uniform picking density, the probability $p_i$ of picking in aisle $i$ equals the length of that aisle divided by the sum of lengths of all picking aisles. For an aisle in the upper region,

$$p^u_i = \frac{l^u_i}{h - w + 2(\sum_{k \in U} l^u_k + \sum_{j \in L} l^h_j)}.$$  

For an aisle in the lower region,

$$p^\ell_j = \frac{l^h_j}{h - w + 2(\sum_{i \in U} l^u_i + \sum_{k \in L} l^h_k)}.$$  

Expected travel cost for a pick in a fishbone warehouse is comprised of a term for the center (vertical) aisle, plus two times the terms for upper and lower aisles (to account for both half-spaces),

$$E[C] = p^u_0 E[C_0] + 2 \left( \sum_{i \in U} p^u_i E[C^u_i] + \sum_{j \in L} p^\ell_j E[C^\ell_j] \right). \tag{5}$$  

The optimization problem is to choose $b$ such that $E[C]$ is minimized, subject to $0 \leq b \leq h - w$.

The problem has only one variable, but the objective function can be shown by counterexample not to be convex. Therefore, we apply numerical non-linear optimization techniques, as before. For practical purposes, it is easy to run the model, then inspect a plot of the objective function value as a function of $b$ to ensure the solution is not a local optimum. (It would also be possible to require $b$ to take on integer values, and then enumerate the solution space.)

### 3.2 Fishbone Designs

For example solutions, we assume standard pallet sizes and aisle widths as before, and we measure distances in units of pallets. For a fishbone design we must consider the value of $w$ carefully, because picking aisles extend from the cross aisle at different orientations. This is easily adjusted in the model.
Figure 6: A warehouse with fishbone aisles. Expected travel distance to a pick in this
warehouse is 20.3 percent lower than in a traditional warehouse with the same total length of
picking aisles (although the fishbone design occupies 3.0% more space). An implementation
of this design in practice would likely clear away some storage racks near the P&D point.

Example. Figure 6 shows the solution for the equivalent of 21 vertical picking aisles, each
having length of 50 pallet locations. In this case, the optimal $b = h - w$. The intuition is
that such designs have a cross aisle that cuts through the “middle” of the picking space, and
therefore, because every pick uses the cross aisle, it should be centrally placed. We have
solved the fishbone design problem for warehouses of many sizes, and the optimal $b = h - w$
in almost every case. The exceptions are among warehouses with relatively few, very long
vertical aisles. The optimal $b$ for these warehouses is slightly less than $h - w$.

It is important to recognize that as $b$ increases and the slope of the cross aisle increases, the
total length of picking aisles decreases slightly (see Figure 7). Therefore, we must be careful
when interpreting a solution. This manifests an important tradeoff between storage density and expected retrieval distance, which we must consider when comparing fishbone designs with a traditional warehouse. To do so, we first model a fishbone warehouse, then model a traditional warehouse with the same number of vertical aisles, with the length of the picking aisles adjusted so that the total aisle length equals that of the fishbone design. The design in Figure 6 has expected travel cost 20.3 percent lower than an equivalent traditional warehouse. Figure 8 shows how the potential travel cost advantage varies for different configurations. In practice, unit-load warehouses range in size between 20 and 40 aisles.

4 Bounds

Warehouses with V-shaped cross aisles or fishbone aisles offer a significant potential savings in retrieval time, but how close are they to optimal? Here we offer a lower bound on travel distance based on an imaginary warehouse in which items are distributed uniformly and continuously throughout the picking space, and workers can “fly” directly to and from any
Figure 8: Percent improvement of a fishbone aisle design over a traditional design for several configurations.

location. We use the bound to compute the maximum possible improvement of any design over a traditional warehouse, in which travel is rectilinear.

Without loss of generality, we can consider a rectangular half-warehouse with dimensions \( a \times b \) and a single P&D point in the bottom-left corner. Picking activity is distributed continuously and uniformly throughout the space. We used Mathematica to compute the minimum expected distance from a point in the rectangle to the P&D point,

\[
\frac{1}{ab} \int_0^a \int_0^b \sqrt{x^2 + y^2} \, dy \, dx = \frac{1}{6ab} \left( 2ab\sqrt{a^2 + b^2} + a^3 \log \left( \frac{b + \sqrt{a^2 + b^2}}{a} \right) + b^3 \log \left( \frac{a + \sqrt{a^2 + b^2}}{b} \right) \right). 
\]

If travel is rectilinear, as in a traditional warehouse, expected distance would be,

\[
\frac{1}{ab} \int_0^a \int_0^b x + y \, dy \, dx = \frac{1}{2} (a + b).
\]

For a square picking half-space with unit-length sides, which has the intuitive appeal of being “balanced” (Francis, 1967, showed this to be an optimal shape, under certain assumptions), an imaginary warehouse has expected distance to make a pick,

\[
\int_0^1 \int_0^1 \sqrt{x^2 + y^2} \, dy \, dx = \frac{1}{3} \left( \sqrt{2} + \sinh^{-1}(1) \right) \approx 0.7652,
\]
If travel is rectilinear, as in a traditional warehouse, expected distance is
\[ \int_0^1 \int_0^1 x + y \, dy \, dx = 1. \]

The implication is that, if picking activity is distributed “approximately uniformly” in a square picking half-space, the best possible aisle design can offer no more than \(1 - 0.7652 \approx 23.5\) percent reduction in expected travel distance. (It is worthwhile noting that a traditional unit-load warehouse—with or without cross aisles—performs as badly as possible, when average distance is the performance metric and workers travel to picks along shortest paths.)

The result is different for non-square picking half-spaces. Figure 9 illustrates the benefit of direct, “travel-by-flight” over the rectilinear travel required in a traditional warehouse as a function of the ratio of width to height of the picking space. The benefit has its maximum at 2:1, which corresponds to the commonly found square half-space that conforms to the results in Francis (1967). As the ratio increases, the advantage of direct travel decreases.

The implication is that the potential advantage of better aisle designs is greatest for the warehouse sizes most common in industry. Moreover, the 23.5 percent potential improvement with respect to a theoretical warehouse space suggests that the fishbone design, which showed a 20.3 percent improvement for a square half-warehouse, is close to optimal (Figure 9 also shows comparable results for V-shaped aisles). Example travel paths represented in Figure 10 show why this is the case: fishbone designs are particularly good for accessing locations near a 45-degree diagonal from the P&D point, the same locations for which rectilinear travel is especially poor.

5 Implementation Issues and Conclusions

We have relaxed only two of several unspoken design rules of previous research and practice, and have found that there is much to be gained by considering new aisle configurations for unit-load warehouses. Designs we develop in this paper could be useful in any unit-load warehouse with a single, dominant P&D point.
Figure 9: Benefit of direct travel over rectilinear travel to and from the P&D point as a function of the ratio of width to height of the warehouse (labeled “Bound”). Dots under the curve represent improvement offered by fishbone warehouses and warehouses with V-shaped cross aisles.

Figure 10: An illustration of why travel along fishbone aisles is close to optimal: solid black lines represent fishbone travel, the dashed line rectilinear travel, the gray line direct “travel by flight.”
Our results suggest that, for unit-load warehouses, new aisle designs could lead to higher throughput, or to significantly reduced costs of picking. For new warehouses, designers must weigh the value of reduced operating costs of these designs with the fixed cost of needing a slightly larger warehouse. In some situations, bulk areas are seldom visited, and a focus on density rather than retrieval cost is appropriate; but for many applications, labor costs are high enough to justify new designs.

We offer two new designs: The first inserts a V-shaped cross aisle (with both segments slightly curved) into a warehouse with parallel picking aisles. Contrary to the established orthodoxy of unit-load warehouse design, inserting a cross aisle with this shape reduces the expected travel distance for workers, and that by about 8–12 percent, depending on dimensions of the warehouse. The second design has “fishbone” picking aisles which extend horizontally and vertically from diagonal (“spine”) cross aisles. We showed this design can reduce expected travel cost by more than 20 percent, and that this is close to optimal.

If fishbone designs offer lower expected retrieval distances than designs with V-shaped cross aisles, why even consider the latter? Because, they have several advantages:

- Access into and out of the space is easier, which is a benefit if workers occasionally must make trips to or from points other than the P&D point.

- Forklift traffic is distributed over two cross aisles (diagonal and bottom) in a design with V-shaped cross aisles, unlike in the traditional warehouse and the fishbone warehouse, where all traffic is concentrated along one cross aisle.

- Workers are less likely to become disoriented in a V-shaped cross aisle warehouse because it is similar to a traditional warehouse. For an industry in which worker turnover is high and experience is low, this can be an important benefit.

- The numbering scheme for picking locations is more intuitive than it would be for a fishbone warehouse.

- It is possible to retrofit an existing traditional warehouse with a V-shaped cross aisle,
simply by removing appropriate portions of rack. On this point, we acknowledge the probably insurmountable psychological barrier of paying money to remove “good storage locations.” Nevertheless, it is possible, and should be considered if storage capacity allows and there is a significant operational advantage.

We should reiterate that our designs apply only to unit-load warehouses with a single, centrally-located P&D point, and are based on an assumption of uniformly distributed activity within and among picking aisles. Interesting extensions include models for case-picking warehouses, in which workers visit multiple locations per tour, and for warehouses with multiple P&D points.

Appendix

**Proposition 1** If $w = 0$, an optimal solution has $b_0 = 0$; that is, the optimal cross aisle in a picking half-space begins at the P&D point.

**Proof.** By contradiction. Assume there exists an optimal cross aisle in a picking half-space that intersects aisle 0 at a point $b_0 > 0$. Any pick that uses the cross aisle requires $b_0 + \sqrt{a^2 + (b_1 - b_0)^2}$ travel to get to the first aisle, which is greater than $\sqrt{a^2 + b_1^2}$ by the triangle inequality; that is, letting $b_0 = 0$ reduces the expected distance to any pick using the cross aisle, without changing the distance to any pick not using it. Therefore, the cross aisle is not optimal, a contradiction. \qed

**Proposition 2** If $w = 0$, an optimal solution has $b_i \geq b_{i-1}$ for $i \geq 1$.

**Proof.** Assume there is an optimal solution with the $b_i$’s not sorted from smallest to greatest. Construct a modified warehouse with the aisles resequenced such that the $b_i$’s are non-decreasing. For every aisle $i$, the distance $\sum_{k=1}^{i} d_k$ along the cross aisle to the picking aisle in the modified warehouse is no greater than in the original, optimal warehouse. Moreover, for at least one aisle it is less.
Now consider travel within the picking aisles. Total travel to points above the cross aisle is the same for both warehouses, because the \( b_i \)'s are the same, only resequenced. However, travel below the cross aisle is different because some \( q_i \)'s are different. Observe that the region between \( q_i \) and \( b_i \) in an aisle represents, by definition, that part of the picking aisle to which it is cheaper to travel along the cross aisle than along the bottom aisle; therefore, \( \sum_i (b_i - q_i) \) represents the total portion of the warehouse below the cross aisle that benefits from the cross aisle.

Consider,

\[
\sum_{i=1}^{n} (b_i - q_i) = \sum_{i=1}^{n} \left[ b_i - \frac{1}{2} \left( \sum_{k=1}^{i} d_k + b_i - ia \right) \right] = \sum_{i=1}^{n} b_i - \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{k=1}^{i} d_k + \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} ia \right),
\]

which is greater for the modified warehouse because \( \sum_{i=1}^{n} \sum_{k=1}^{i} d_k \) is less than in the optimal warehouse, and all other terms are the same. Therefore, the modified warehouse has lower total travel costs to picking aisles and lower total travel costs within picking aisles, a contradiction. \( \square \)

**Proposition 3** If \( w = 0 \), the optimal point of intersection \( b_i \) satisfies \( b_i \leq \frac{2}{3} h + \frac{1}{3} \left( \sum_{k=1}^{i} d_k - ia \right) \).

**Proof.** Observe,

\[
\begin{align*}
b_i &\leq \frac{2}{3} h + \frac{1}{3} \left( \sum_{k=1}^{i} d_k - ia \right) \\
\frac{4}{3} b_i &\leq \frac{2}{3} h + \frac{1}{3} \left( b_i + \sum_{k=1}^{i} d_k - ia \right) \\
b_i &\leq \frac{1}{2} \left( h + \frac{1}{2} \left[ b_i + \sum_{k=1}^{i} d_k - ia \right] \right) \\
&= \frac{1}{2} (h + q_i),
\end{align*}
\]

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Therefore, it is sufficient to prove that \( b_i \leq \frac{1}{2}(h + q_i) \), which we do by induction.

The claim is true for aisle zero by Proposition 1. Assume it is true for aisle \( k \), and consider aisle \( k + 1 \): If the claim is not true for \( k + 1 \), then \( b_{k+1} > (h + q_{k+1})/2 \) in an optimal solution. Notice that there are three variable components of the expected cost for a pick in aisle \( k + 1 \): distance along the cross aisle, expected within-aisle distance for a pick above \( q_{k+1} \), and expected within-aisle distance for a pick below \( q_{k+1} \). The latter two components are weighted according to the length of the respective regions.

By setting \( b_{k+1} = (h + q_{k+1})/2 \) we reduce all three: (1) the travel distance \( d_{k+1} \) to the aisle is lower, because \( b_k \leq (h + q_k)/2 \); (2) the weighted expected within-aisle distance to a pick above \( q_{k+1} \) is lower, because \( b_{k+1} = (h + q_{k+1})/2 \) (the segment midpoint) minimizes this quantity; and (3) the weighted expected within-aisle distance to a pick below \( q_{k+1} \) is lower, because this region is now smaller. By a similar argument, we can reduce the cost of picking in all subsequent aisles. \( \square \)

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